Anuran Makur
19 June 2016

Circular Law:

History: ★ We show these.

- Dyson ★ J. Ginibre (1965): Ginibre ensemble joint eigenvalue density formula (shorter proof due to Dyson – see P. 582 of Mehta, Random Matrices)
- ★ M.L. Mehta (1967): Proof of mean version of circular law for $\mathbb{C}$ Gaussian ensemble using Ginibre formula
- in part ★ J.W. Silverstein (1984): Proof of a.s. version of circular law of $\mathbb{C}$ Gaussian ensemble [private comm. in Hwang '86]
- V.L. Girko (1984): Non-rigourous proof of "universal" circular law
- A. Edelman (1997): Proof of circular law for $\mathbb{R}$ Gaussian ensemble
- Z. D. Bai (1997): Rigorous "universal" circular law with bounded density/moment assumptions
- P. Śniady (2002): Free probability version of circular law with Brown measure
- T. Tao & V. Vu (2010): Complete proof of "universal" circular law

lots of work

Dyson's Proof of Ginibre Formula:

Let $M_n \in \mathbb{C}^{n\times n}$ be an iid $\mathcal{N}_{\mathbb{C}}(0,1)$ matrix.

<u>Thm:</u> (Ginibre) The joint eigenvalue density is given by:

$$p_n(\lambda_1,\dots,\lambda_n) = \underset{\text{constant}}{C_n} \underbrace{\prod_{i<j} |\lambda_i-\lambda_j|^2}_{=|\Delta(\lambda_1,\dots,\lambda_n)|^2} \exp\left(-\sum_{i=1}^{n}|\lambda_i|^2\right), \quad \forall \lambda_1,\dots,\lambda_n \in \mathbb{C}.$$

↑ Vandermonde determinant

<u>Proof:</u> (Dyson)

First observe that all eigenvalues of M_n are distinct a.s.. So, M_n is diagonalizable: $M_n = PDP^{-1}$ (P full rank, D diagonal). Then, $M_n = QTQ^H$ [Schur decomp.] (Q unitary, T upper Δ).

Let $P = \underbrace{Q}_{\text{unitary}}\underbrace{R}_{\text{upper }\Delta}$ & $P^{-1} = \underbrace{R^{-1}}_{\text{upper }\Delta}Q^H$ be QR decompositions.

$dM_n = dQ\,T\,Q^H + Q\,dT\,Q^H + Q\,T\,dQ^H$ and $\underbrace{Q^H dQ = -dQ^H Q}_{\text{anti-Hermitian}}$

$$\Rightarrow \underbrace{Q^H dM_n Q}_{\triangleq dA} = Q^H dQ\,T + dT + T\,dQ^H Q$$

$$= dT + Q^H dQ\,T - T\underbrace{Q^H dQ}_{\triangleq dH} \quad \text{with } (dH)^H = -dH$$

$$\Rightarrow dA = dT + (dH)T - T(dH), \text{ where } \underline{(dH)_{ii} = 0, \forall i}.$$

Hence, we have:

$$(dA)_{jk} = \underbrace{(dT)_{jk}}_{=0 \text{ for } j>k} + (T_{kk}-T_{jj})(dH)_{jk} + \sum_{\ell<k}(dH)_{j\ell}T_{\ell k} - \sum_{\ell>j}T_{j\ell}(dH)_{\ell k}, \quad 1\leq j,k\leq n.$$

Order the indices (j,k), $1\leq j,k\leq n$, so that (j_1,k_1) precedes (j_2,k_2) if $j_1>j_2$ or $j_1=j_2, k_1<k_2$. If $j\leq k$, we take T_{jk} as variable & if $j>k$, we take $H_{j,k}$.

Degrees of freedom:

M_n – $2n^2$ $\mathbb{R}$ dof

T – $n(n+1)$ $\mathbb{R}$ dof

Q – $\underbrace{2n^2}_{\mathbb{R}\text{ dof in entries}} - \underbrace{n(n-1)}_{\perp\text{-constraints}} - n = n^2$ $\mathbb{R}$ dof

$|\cdot|=1$ is $\mathbb{R}$ constraint

$\perp$-constraints require $|\cdot|$ & phase $\Rightarrow \mathbb{C}$ constraints $= \frac{n(n-1)}{2}$

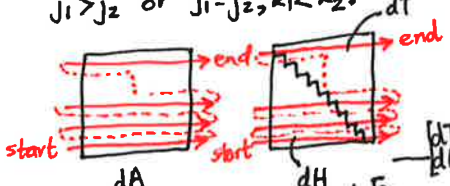
$\Rightarrow (Q,T)$ have n extra dof.

<u>Why?</u> Because $M_n = \underbrace{QV}_{Q'}\underbrace{V^H T V}_{T'}\underbrace{V^H Q^H}_{Q'^H}$

where V = diagonal with $e^{i\theta_k}$'s.

↑ extra n $\mathbb{R}$ dof.

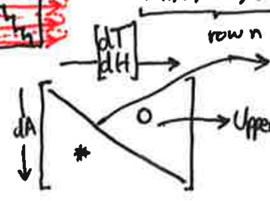
→ Use <u>n dof to make Im part of $(Q^H dQ)_{ii} = 0$</u> for every i.



$$\underbrace{A_{n,1}, A_{n,2},\dots,A_{n,n}}_{\text{row } n}, \underbrace{A_{n-1,1},\dots,A_{n-1,n}}_{\text{row } n-1},\dots,\underbrace{A_{1,1},\dots,A_{1,n}}_{\text{row } 1}$$

$$\underbrace{H_{n,1},\dots,H_{n,n-1},T_{n,n}}_{\text{row } n}, \underbrace{H_{n-1,1},\dots,H_{n-1,n-2},T_{n-1,n-1},T_{n-1,n}}_{\text{row } n-1},\dots,\underbrace{T_{1,1},\dots,T_{1,n}}_{\text{row } 1}$$

Jacobian matrix =



→ Diagonal $= \underset{\text{wrt }(dH)_{jk}}{T_{kk}-T_{jj}}$ or $\underset{\text{wrt }(dT)_{jk}}{1}$

→ Upper Diagonal $= 0$

[$(dA)_{jk}$ only dep.s on $(dH)_{rs}$ with $r\geq j, s\leq k$ (i.e. preceding $(dH)_{rs}$'s) or/and $(dT)_{jk}$]

$\Rightarrow$ Jacobian determinant $= \left|\prod_{j>k} T_{kk}-T_{jj}\right|^2$

$\mathbb{C}$ Jacobian has $|\cdot|^2$

[continued.]

Anuran Makur

Note: $\underbrace{(A \otimes B)}_{n\times n\ \mathbb{C}\text{ matrices}} X \triangleq B X A^H$

Jacobian $= |\det(A\otimes B)|^2$
$= |\det((A\otimes I)(I\otimes B))|^2$
$= |\det(A)^n \cdot \det(B)^n|^2$

Proof cont'd:

We have: $(dA)^\wedge = \prod_{j<k} |T_{jj} - T_{kk}|^2 \, (dT)^\wedge \underbrace{(Q^H dQ)^\wedge}_{=(dH)^\wedge}$. ← wedge product

Notice that if $A = Q^H M Q = (Q^H \otimes Q^H) M$, then its Jacobian is:

$|\det(Q^H \otimes Q^H)|^2 = \left(|\det(Q)|^{2n}\right)^2 = \underline{\underline{1}}.$

So, $dA = Q^H dM Q$ and $(dA)^\wedge = (\text{Jacobian})(Q^H dM Q)^\wedge = (Q^H dM Q)^\wedge$.

$$\Rightarrow \underbrace{(Q^H dM_n Q)^\wedge}_{dA} = \boxed{(dM_n)^\wedge = \prod_{j<k} |T_{jj} - T_{kk}|^2 \, (dT)^\wedge (Q^H dQ)^\wedge}.$$

eigenvalues

Since the joint density of M_n is: $C_n \exp(-\|M\|_{Fro}^2)\,(dM)^\wedge$, we have:
↑ constant

$$C_n \exp(-\|M\|_{Fro}^2)(dM)^\wedge = C_n \exp(-\|QTQ^H\|_{Fro}^2) \prod_{j<k} |T_{jj} - T_{kk}|^2 (dT)^\wedge (Q^H dQ)^\wedge$$
$$= C_n \exp(-\|T\|_{Fro}^2) \prod_{j<k} |\underbrace{T_{jj}}_{\triangleq \lambda_j} - T_{kk}|^2 (dT)^\wedge (Q^H dQ)^\wedge$$

← jth eigenvalue of M_n

$$= C_n \left[\exp\left(-\sum_{i=1}^n |\lambda_i|^2\right) \prod_{j<k} |\lambda_j - \lambda_k|^2 \, d\lambda_1 \cdots d\lambda_n\right]\left[\exp(-\|\hat{T}\|_{Fro}^2)(d\hat{T})^\wedge (Q^H dQ)^\wedge\right]$$

$\hat{T}$ = strictly upper $\triangle$ part of T

$$\Rightarrow \boxed{p_n(\lambda_1, \ldots, \lambda_n) \propto \prod_{j<k} |\lambda_j - \lambda_k|^2 \exp\left(-\sum_{i=1}^n |\lambda_i|^2\right)}.$$

▨